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Asymptotic Behavior of Orthogonal Polynomials Corresponding to Measure with Discrete Part off the Unit Circle

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For a positive measure μ on the unit circle (Γ) in the complex plane, m points z_j off Γ and m positive numbers A_j, $j = 1, 2, \ldots, m$, we investigate the asymptotic behavior of orthonormal polynomials $\Phi_n(z)$ corresponding to $d\mu/2\pi + \sum_{j=1}^{m} A_j \delta_{z_j}$, where δ_z denotes the unit measure supported at point z. Our main result is the relative asymptotics of $\Phi_n(z)$ with respect to the orthonormal polynomial corresponding to $d\mu/(2\pi)$ off and on Γ. © 1994 Academic Press, Inc.

1. INTRODUCTION

Let μ be a finite positive measure with an infinite set as support on the unit circle $\Gamma := \{z \in \mathbf{C} \mid |z| = 1\}$. Such a measure can be represented by a nondecreasing function μ on $[0, 2\pi)$. Denote by $\mathscr{P}_n$ the set of polynomials of degree at most n. Let $\varphi_n(z) = \kappa_n z^n + \cdots \in \mathscr{P}_n$ ($\kappa_n > 0$) be the nth orthonormal polynomial corresponding to $\mu/(2\pi)$ on Γ, i.e.,

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi_n(z) \overline{z^k} \, d\mu(\theta) = \kappa_n^{-1} \delta_{kn}, \qquad k = 0, 1, \ldots, n, \; z = e^{i\theta}.$$

Suppose $z_1, z_2, \ldots, z_m$ are m fixed points outside Γ. For m positive numbers $A_1, A_2, \ldots, A_m$, construct $\nu = \mu/(2\pi) + \sum_{j=1}^{m} A_j \delta_{z_j}$, where δ_z denotes the (Dirac delta) unit measure supported at point z. Let $\Phi_n(z) = \gamma_n z^n + \cdots \in \mathscr{P}_n$ ($\gamma_n > 0$) be the (unique) nth orthonormal polynomial

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corresponding to ν on $\Gamma \cup \{z_1, z_2, \dots, z_m\}$, i.e.,

$$\frac{1}{2\pi}\int_0^{2\pi}\Phi_n(z)\overline{z^k}\,d\mu(\theta) + \sum_{j=1}^{m} A_j\Phi_n(z_j)\overline{z_j^k} = \gamma_n^{-1}\delta_{kn},$$

$$k = 0, 1, \dots, n,\ z = e^{i\theta}.$$

Note that both $\Phi_n(z)$ and γ_n depend on A_j, $j = 1, 2, \dots, m$, although this dependence is not explicitly given in our notation.

The purpose of this paper is to study the asymptotic behavior of $\Phi_n(z)$ off and on Γ under fairly mild assumptions on μ. The asymptotics of $\varphi_n(z)$ has been investigated extensively (see, for example, [3, 5, 10, 15, 17]). To take this advantage in our study of $\Phi_n(z)$, we compare $\Phi_n(z)$ with $\varphi_n(z)$ in the form of relative asymptotics [11]; i.e., we investigate $\Phi_n(z)/\varphi_n(z)$.

For measures supported on the real line **R**, the comparison of orthonormal polynomials corresponding to measures differed by a discrete part (on **R**) has been studied and used in Nevai's monograph [13] (especially, see Thm. 25 and 26 on p. 136 in [13]). More recently, asymptotics of orthogonal polynomials (or the L_p extremal monic polynomials) corresponding to measure (on **C**) with discrete part off a curve or an arc in the complex plane (under Szegő's condition on the measure μ) have been considered by Kalyagin [7], Kaliaguine [6] and Kaliaguine and Benzine [8]. Here we take a nice curve, the unit circle, and relax the assumption on the measure. The special case when $m = 1$ has been studied by Cachafeiro and Marcellan. Some of our results have been established already by Cachafeiro and Marcellan ([1, 2]) for that case. Our particular interest in the unit circle case lies in its potential application to rational approximation.

The main results are stated in Section 2, and their proofs are given in Section 4. Section 3 is devoted to the lemmas needed for the proof in Section 4. Finally, we will discuss some relevant results (zero location and distribution, etc.) in Section 5.

2. Main Results

We say measure μ belongs to class N (denoted by $\mu \in N$) if

$$\lim_{n\to\infty}\frac{\varphi_n(0)}{\kappa_n} = 0.$$

Class N of measures on Γ is analogous to the Nevai class $M(0,1)$ of measures on **R** [13]. It is well known, by a theorem of Rahmanov (cf. [15] or [10]), that condition $\mu'(\theta) > 0$ a.e. on Γ implies $\mu \in N$.

Let

$$B(z) := \prod_{j=1}^{m} (z - z_j)/(1 - \bar{z}_j z), \tag{1}$$

and $\lambda := |B(0)|/B(0)$.

We first discuss the ratio of the two leading coefficients.

THEOREM 1. *If $\mu \in N$, then*

$$\lim_{n\to\infty} \frac{\gamma_n}{\kappa_n} = \prod_{j=1}^{m} \frac{1}{|z_j|}. \tag{2}$$

Furthermore, the above limited process is locally uniform for $A_j > 0$, $j = 1, 2, \ldots, m$, in the sense that, for any $\delta > 0$ and $\varepsilon > 0$, there exists an integer $N = N(\delta, \varepsilon) > 0$ such that

$$\left| \frac{\gamma_n}{\kappa_n} - \prod_{j=1}^{m} \frac{1}{|z_j|} \right| < \varepsilon,$$

for all $n \geq N$ and $A_j \geq \delta$, $j = 1, 2, \ldots, m$.

Remark 1. It is easy to check that

$$\prod_{j=1}^{m} \frac{1}{|z_j|} = \frac{1}{|B(0)|} = \lambda B(\infty),$$

where $B(\infty) := \lim_{z\to\infty} B(z)$.

Remark 2. Note that the limit value in (2) is independent of $\{A_j\}_{j=1}^{m}$, while the limit process is locally uniform in $A_j > 0$, $j = 1, 2, \ldots, m$.

For the ratio of the two orthonormal polynomials, we have

THEOREM 2. *If $\mu \in N$, then*

$$\lim_{n\to\infty} \frac{\Phi_n(z)}{\varphi_n(z)} = \lambda B(z), \tag{3}$$

uniformly for $|z| \geq 1$. Furthermore, the limit process is locally uniform for $A_j > 0$, $j = 1, 2, \ldots, m$, in the same sense as described in Theorem 1.

Remark 3. From the known results on the asymptotics of $\varphi_n(z)$, we can use Theorem 2 to obtain the corresponding results for $\Phi_n(z)$. For example, we can have the ratio asymptotics of $\Phi_n(z)$ from that of $\varphi_n(z)$ (cf. (7) in Section 2), and if we make a stronger assumption on μ (say Szegő's condition), then we can get better result. We leave the formulation of these results to the reader.

The following result tells us that $\Phi_n(z_j)$ $(j = 1, 2, \ldots, m)$ tends to zero geometrically fast.

COROLLARY 3. *If $\mu \in N$, then, for each $\varepsilon \in (0,1)$,*

$$\lim_{n \to \infty} z_j^{\varepsilon n} \Phi_n(z_j) = 0, \qquad j = 1, 2, \ldots, m.$$

This result will make one guess that $\Phi_n(z)$ behaves like $Q(z)\prod_{j=1}^{m}(z - z_j)$ for some $Q(z)$ as $n \to \infty$ in certain sense. (In Section 5, we shall discuss more on location and distribution of the zeros of Φ_n as $n \to \infty$.) On the other hand, it is proved in [9] that for each *fixed* n, we have

$$\lim_{\substack{A_j \to \infty \\ j = 1, 2, \ldots, m}} \Phi_{n+m}(z) = \Psi_n(z) \prod_{j=1}^{m} (z - z_j), \tag{4}$$

locally uniformly for $z \in \mathbf{C}$, where $\Psi_n(z) = \beta_n z^n + \cdots \in \mathscr{P}_n$ is the nth orthonormal polynomial corresponding to measure $|\Pi_{j=1}^{m}(z - z_j)|^2 \, d\mu$ on Γ, i.e.,

$$\frac{1}{2\pi} \int_0^{2\pi} \Psi_n(z) \overline{z^k} \left| \prod_{j=1}^{m} (z - z_j) \right|^2 d\mu(\theta) = \beta_n^{-1} \delta_{kn},$$

$$k = 0, 1, \ldots, n, \; z = e^{i\theta}.$$

It is useful to get the asymptotics of such orthonormal polynomials. Now, because of the uniformity in A_j's in Theorems 1 and 2, we can obtain the relative asymptotics of $\Psi_n(z)$ with respect to $\varphi_n(z)$.

THEOREM 4. *If $\mu \in N$, then there holds*

$$\lim_{n \to \infty} \frac{\Psi_n(z)}{\varphi_n(z)} = \frac{\lambda z^m}{\prod_{j=1}^{m} (1 - \bar{z}_j z)}, \tag{5}$$

uniformly for $|z| \geq 1$.

This result turns out to be a special case of theorems in [11]. We hope this will shed some light on the general situation.

So far we have always assumed the isolated mass points are all off Γ, i.e., $|z_j| > 1$, $j = 1, 2, \ldots, m$. If we just look for asymptotics outside Γ

then we need only assume $|z_j| \geq 1$ for $j = 1, 2, \ldots, m$. For example, we have:

COROLLARY 5. *If* $\mu \in N$ *and* $|z_j| \geq 1$ $(j = 1, 2, \ldots, m)$, *then*

$$\lim_{n \to \infty} \frac{\gamma_n}{\kappa_n} = \prod_{|z_j| > 1} \frac{1}{|z_j|},$$

and

$$\lim_{n \to \infty} \frac{\Phi_n(z)}{\varphi_n(z)} = \prod_{|z_j| > 1} \frac{z - z_j}{1 - \bar{z}_j z} \cdot \frac{|z_j|}{-z_j},$$

uniformly for every compact subset of $|z| > 1$.

Here the empty product is defined as of value 1.

3. LEMMAS

We need to introduce some notation. The *-transform $P_n^*(z)$ of polynomial $P_n(z)$ of degree n is defined as $P_n^*(z) = z^n \overline{P_n(1/\bar{z})}$. One can check that, for $z \in \Gamma$, $|P_n^*(z)| = |P_n(z)|$. The reproducing kernel function K_n is defined by

$$K_n(z; \zeta) = \sum_{k=0}^{n-1} \overline{\varphi_k(z)} \varphi_k(\zeta),$$

and by the Christoffel–Darboux formula (cf. [5, p. 3]) we have

$$K_n(z; \zeta) = \frac{\overline{\varphi_n^*(z)} \varphi_n^*(\zeta) - \overline{\varphi_n(z)} \varphi_n(\zeta)}{1 - \bar{z}\zeta}.$$

In the following, for a set S, we will say "locally uniformly for S" to mean "uniformly for every compact subset of S."

LEMMA 1. *If* $\mu \in N$, *then*

$$\lim_{n \to \infty} \frac{K_n(z; \zeta)}{\overline{\varphi_n(z)} \varphi_n(\zeta)} = \frac{1}{\bar{z}\zeta - 1},$$

locally uniformly for $|z| > 1$, *and* $|\zeta| > 1$.

Proof. It is proved in [11, Thm. 4 and its proof] that $\mu \in N$ implies

$$\lim_{n \to \infty} \frac{\varphi_n^*(z)}{\varphi_n(z)} = 0, \tag{6}$$

locally uniformly for $|z| > 1$. The lemma then follows from the Christoffel–Darboux formula. ∎

LEMMA 2. *If $\mu \in N$, then*

$$\lim_{n \to \infty} \frac{\varphi_{n+1}(z)}{\varphi_n(z)} = z, \tag{7}$$

uniformly for compact sets of $|z| \geq 1$. Consequently, for each $\varepsilon \in (0,1)$,

$$\lim_{n \to \infty} \frac{|\varphi_n(z)|}{|z|^{\varepsilon n}} = \infty, \tag{8}$$

locally uniformly for $|z| > 1$.

Proof. Formula (7) is well known (cf. [10] or [15]). For (8), let $r > 1$ and $\delta \in (\varepsilon, 1)$, then by (7), there exists an integer $L > 0$ such that

$$\left|\frac{\varphi_{n+1}(z)}{\varphi_n(z)}\right| \geq \frac{1}{r}|z|,$$

for all $n \geq L$, and $|z| \geq r^{1/(1-\delta)}$. So, for $n \geq L$ and $|z| \geq r^{1/(1-\delta)}$,

$$\left|\frac{\varphi_{n+1}(z)}{\varphi_L(z)}\right| = \prod_{k=L}^{n} \left|\frac{\varphi_{k+1}(z)}{\varphi_k(z)}\right| \geq \left(\frac{|z|}{r}\right)^{n-L+1} \geq |z|^{\delta(n-L+1)}.$$

Now formula (8) follows from the above inequalities and the fact that $\varphi_L(z) \neq 0$ for $|z| \geq 1$. ∎

LEMMA 3. *For all $n \geq 0$, there holds*

$$\frac{\gamma_n}{\kappa_n} \leq 1.$$

Proof. This lemma is the consequence of the extremality of the monic polynomial $\kappa_n^{-1}\varphi_n(z)$ (cf. [17, Thm. 11.1.2, p. 289]):

$$\frac{1}{\kappa_n^2} = \min_{p \in \mathscr{P}_{n-1}} \frac{1}{2\pi} \int |z^n + p(z)|^2 \, d\mu$$

$$\leq \frac{1}{2\pi} \int \left|\frac{\Phi_n(z)}{\gamma_n}\right|^2 d\mu \leq \frac{1}{\gamma_n^2} \int |\Phi_n(z)|^2 \, d\nu = \frac{1}{\gamma_n^2}. \quad ∎$$

LEMMA 4. *For m distinct points $z_1, z_2, \ldots, z_m$ outside the unit circle, the matrix*

$$\mathbf{T}_m := \left(\frac{1}{\bar{z}_j z_k - 1} \right)_{k,j=1}^{m}$$

is non-singular.

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_m)^t \in \mathbf{C}^m$, and $\mathbf{T}_m \mathbf{x} = \mathbf{0}$, then we need to show $\mathbf{x} = \mathbf{0}$. To do so, define

$$F(z) := \sum_{j=1}^{m} \frac{x_j}{\bar{z}_j z - 1}.$$

Then $F(z)$ can be written as $p(z)/\prod_{j=1}^{m}(\bar{z}_j z - 1)$ with $p \in \mathscr{P}_{m-1}$. However, $\mathbf{T}_m \mathbf{x} = \mathbf{0}$ implies $F(z_k) = 0$, $k = 1, 2, \ldots, m$, and so $p(z_k) = 0$, $k = 1, 2, \ldots, m$; hence $p(z) \equiv 0$. Thus $F(z) \equiv 0$, which implies that $\mathbf{x} = \mathbf{0}$ since $\{1/(\bar{z}_j z - 1)\}_{j=1}^{m}$ forms a set of linearly independence function set. ∎

LEMMA 5. *For m distinct points $z_1, z_2, \ldots, z_m$ outside the unit circle, let $B(z)$ be defined as in* (1), *then there exist a unique set of non-zero complex numbers $r_1, r_2, \ldots, r_m$ such that*

$$B(z) = \frac{1}{\overline{B(0)}} + \sum_{j=1}^{m} \frac{r_j}{1 - \bar{z}_j z}. \tag{9}$$

Proof. The existence of the above partial fraction representation of $B(z)$ is obvious. The uniqueness follows from the linear independence of the set $\{(1 - \bar{z}_j z)^{-1}\}_{j=1}^{m}$. Finally, that none of the r_j's is zero follows from comparing the poles on both sides of (9). ∎

4. PROOFS OF MAIN RESULTS

Now we are ready to prove our theorems.

Proof of Theorem 1. We need to show the existence of the limit in (2) and calculate the value of the limit.

By Lemma 3, every subsequence of $\{\gamma_n/\kappa_n\}_{n=0}^{\infty}$ contains a convergent subsequence. Let $R \geq 0$ be a limit point of this sequence, and $\Lambda \subseteq \{0, 1, 2, \ldots\}$ satisfy

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} \gamma_n/\kappa_n = R. \tag{10}$$

Note that the orthonormality of $\varphi_n(z)$ yields (on writing $\Phi_n(z) = (\gamma_n/\kappa_n)\varphi_n(z) + p(z)$, with $p \in \mathscr{P}_{n-1}$)

$$\frac{1}{2\pi}\int \Phi_n(z)\overline{\varphi_n(z)}\,d\mu = \frac{\gamma_n}{\kappa_n}.$$

On the other hand, the orthonormality of Φ_n gives (on writing $\varphi_n(z) = (\kappa_n/\gamma_n)\Phi_n(z) + q(z)$, with $q \in \mathscr{P}_{n-1}$)

$$\begin{aligned}\frac{1}{2\pi}\int \Phi_n(z)\overline{\varphi_n(z)}\,d\mu &= \int \Phi_n(z)\overline{\varphi_n(z)}\,d\nu - \sum_{j=1}^{m} A_j \Phi_n(z_j)\overline{\varphi_n(z_j)} \\ &= \frac{\kappa_n}{\gamma_n} - \sum_{j=1}^{m} A_j \Phi_n(z_j)\overline{\varphi_n(z_j)}.\end{aligned}$$

So, we have

$$\frac{\kappa_n}{\gamma_n} - \frac{\gamma_n}{\kappa_n} = \sum_{j=1}^{m} A_j \Phi_n(z_j)\overline{\varphi_n(z_j)}. \tag{11}$$

We now consider the limit behavior of the summation on the right side as $n \to \infty$ and $n \in \Lambda$. Note that $\Phi_n(z) - (\gamma_n/\kappa_n)\varphi_n(z) \in \mathscr{P}_{n-1}$, so according to the reproducing property of the kernel function $K_n(z;\zeta)$ (cf. [17]) and orthogonality of $\varphi_n(z)$ and $\Phi_n(z)$, with $\zeta = e^{i\theta}$,

$$\begin{aligned}&\Phi_n(z) - \frac{\gamma_n}{\kappa_n}\varphi_n(z) \\ &= \frac{1}{2\pi}\int \left(\Phi_n(\zeta) - \frac{\gamma_n}{\kappa_n}\varphi_n(\zeta)\right) K_n(\zeta; z)\,d\mu(\theta) \\ &= \frac{1}{2\pi}\int \Phi_n(\zeta) K_n(\zeta; z)\,d\mu(\theta) = -\sum_{j=1}^{m} A_j \Phi_n(z_j) K_n(z_j; z) \\ &= -\sum_{j=1}^{m}\left\{A_j \Phi_n(z_j)\overline{\varphi_n(z_j)}\varphi_n(z)\frac{(\overline{z_j}z - 1)K_n(z_j; z)}{\overline{\varphi_n(z_j)}\varphi_n(z)}\right\}\frac{1}{\overline{z_j}z - 1},\end{aligned}$$

and so

$$\frac{\Phi_n(z)}{\varphi_n(z)} = \frac{\gamma_n}{\kappa_n} - \sum_{j=1}^{m}\left\{A_j \Phi_n(z_j)\overline{\varphi_n(z_j)}\frac{(\overline{z_j}z - 1)K_n(z_j; z)}{\overline{\varphi_n(z_j)}\varphi_n(z)}\right\}\frac{1}{\overline{z_j}z - 1}. \tag{12}$$

By Lemma 1, we can write

$$A_j \Phi_n(z_j)\overline{\varphi_n(z_j)} \frac{\left(\overline{z_j} z_k - 1\right) K_n(z_j; z_k)}{\overline{\varphi_n(z_j)} \varphi_n(z_k)}$$
$$= A_j \Phi_n(z_j)\overline{\varphi_n(z_j)}(1 + o(1)) =: X_j(1 + o(1)),$$

as $n \to \infty$, uniformly for $j, k = 1, 2, \ldots, m$. On the other hand, since $A_j |\Phi_n(z_j)|^2 \le \int |\Phi_n|^2 \, d\nu = 1$, $j = 1, 2, \ldots, m$, we have

$$\lim_{n \to \infty} \frac{\Phi_n(z_j)}{\varphi_n(z_j)} = 0,$$

by Lemma 2, and the limit is locally uniform for the choice of $A_j > 0$, $j = 1, 2, \ldots, m$. So, letting $z = z_k$, $k = 1, 2, \ldots, m$, in (12) and using the above limit relations, we can obtain

$$\frac{\gamma_n}{\kappa_n} \mathbf{1} = \mathbf{T}_m[\mathbf{X}(1 + o(1))] + o(\mathbf{1}),$$

where $\mathbf{1} := (1, 1, \ldots, 1)^{\mathrm{t}}$, $\mathbf{T}_m$ is defined as in Lemma 4, $\mathbf{X} := (X_1, X_2, \ldots, X_m)^{\mathrm{t}}$, and the first $o(1)$ is independent of A_j's and the second $o(\mathbf{1})$ is locally uniform for $A_j > 0$, $j = 1, 2, \ldots, m$. So, by Lemma 4,

$$\mathbf{X}(1 + o(1)) = \frac{\gamma_n}{\kappa_n} \mathbf{T}_m^{-1} \mathbf{1} + o(\mathbf{1}). \tag{13}$$

However, letting $z = z_k$, $k = 1, 2, \ldots, m$ in (9) will yield

$$\frac{1}{\overline{B(0)}} = \sum_{j=1}^{m} \frac{r_j}{\overline{z_j} z_k - 1}, \qquad k = 1, 2, \ldots, m,$$

i.e.,

$$\mathbf{T}_m (r_1, r_2, \ldots, r_m)^{\mathrm{t}} = \frac{1}{\overline{B(0)}} \mathbf{1},$$

and so

$$\mathbf{T}_m^{-1} \mathbf{1} = \overline{B(0)} (r_1, r_2, \ldots, r_m)^{\mathrm{t}}.$$

Thus, by (13)

$$\mathbf{X}(1 + o(1)) = \frac{\gamma_n}{\kappa_n} \overline{B(0)} (r_1, r_2, \ldots, r_m)^{\mathrm{t}} + o(\mathbf{1}),$$

and so we have, by use of (10),

$$\lim_{\substack{n\to\infty\\ n\in\Lambda}} X_j = \lim_{\substack{n\to\infty\\ n\in\Lambda}} A_j \Phi_n(z_j)\overline{\varphi_n(z_j)} = R\overline{B(0)}r_j, \qquad j = 1,2,\ldots,m. \quad (14)$$

Now letting $n \to \infty$ and $n \in \Lambda$ in (11), we see that R cannot be zero and

$$\frac{1}{R} - R = R\overline{B(0)}\sum_{j=1}^{m} r_j = R\overline{B(0)}\left(B(0) - \frac{1}{\overline{B(0)}}\right).$$

Hence $R = |B(0)|^{-1}$. Since R is an arbitrary limit point of $\{\gamma_n/\kappa_n\}_{k=0}^{\infty}$, we see that the limit $\lim_{n\to\infty}\gamma_n/\kappa_n$ exists and is equal to $|B(0)|^{-1}$. The local uniformity for $A_j > 0$, $j = 1,2,\ldots,m$, and the limit process is evident from our proof. The proof of the theorem is complete. ∎

Proof of Theorem 2. From (12) and the Christoffel–Darboux formula we can write

$$\frac{\Phi_n(z)}{\varphi_n(z)} = \frac{\gamma_n}{\kappa_n} - \sum_{j=1}^{m} A_j\Phi_n(z_j)\overline{\varphi_n(z_j)}\left[\overline{\left(\frac{\varphi_n^*(z_j)}{\varphi_n(z_j)}\right)}\frac{\varphi_n^*(z)}{\varphi_n(z)} - 1\right]\frac{1}{\bar{z}_j z - 1}. \quad (15)$$

So together with (9) this gives for $|z| \geq 1$

$$\left|\frac{\Phi_n(z)}{\varphi_n(z)} - \lambda B(z)\right| \leq \left|\frac{\gamma_n}{\kappa_n} - \frac{\lambda}{\overline{B(0)}}\right| + \sum_{j=1}^{m}\left\{\left|A_j\Phi_n(z_j)\overline{\varphi_n(z_j)} - \lambda r_j\right| + \left|A_j\Phi_n(z_j)\overline{\varphi_n(z_j)}\frac{\varphi_n^*(z_j)}{\varphi_n(z_j)}\right|\right\}\frac{1}{|z_j| - 1},$$

where we have used the fact that $|\varphi_n^*(z)/\varphi_n(z)| \leq 1$ for $|z| \geq 1$. Now using (14) (by Theorem 1, Λ there can be taken as $\{1,2,3,\ldots\}$, $R = |B(0)|^{-1}$, and so the limit values in (14) are λr_j, $j = 1,2,\ldots,m$) and (6) we have

$$\lim_{n\to\infty}\left|\frac{\Phi_n(z)}{\varphi_n(z)} - \lambda B(z)\right| = 0,$$

uniformly for $|z| \geq 1$ and locally uniformly for $A_j > 0$, $j = 1,2,\ldots,m$. ∎

Proof of Corollary 3. The proof follows from (8) in Lemma 2 and (14) in the proof of Theorem 1. ∎

Proof of Theorem 4. Write

$$\frac{\Phi_{n+m}(z)}{\varphi_n(z)} = \frac{\Phi_{n+m}(z)}{\varphi_{n+m}(z)} \frac{\varphi_{n+m}(z)}{\varphi_n(z)},$$

then, by Theorem 2 and (7)

$$\lim_{n\to\infty} \frac{\Phi_{n+m}(z)}{\varphi_n(z)} = \lambda B(z) z^m, \tag{16}$$

uniformly for $|z| \geq 1$ and locally uniformly for $A_j > 0$, $j = 1, 2, \ldots, m$. Now, on letting $A_j \to \infty$ for $j = 1, 2, \ldots, m$ and using (4), we get

$$\lim_{n\to\infty} \frac{\Psi_n(z) \prod_{j=1}^{m} (z - z_j)}{\varphi_n(z)} = \frac{\lambda z^m \prod_{j=1}^{m} (z - z_j)}{\prod_{j=1}^{m} (1 - \bar{z}_j z)},$$

which implies (5). ∎

Proof of Corollary 5. Let us first consider the case when $m = 1$ and $z_1 \in \Gamma$. As in the proof of Theorem 1, by the reproducing property of the kernel function $K_n(z;\zeta)$ and the orthogonality of both $\varphi_n(z)$ and $\Phi_n(z)$.

$$\begin{aligned} \frac{\kappa_n}{\gamma_n}\Phi_n(z) - \varphi_n(z) &= \frac{1}{2\pi}\int \left(\frac{\kappa_n}{\gamma_n}\Phi_n(\zeta) - \varphi_n(\zeta)\right) K_n(\zeta; z)\, d\mu(\theta) \\ &= -\frac{\kappa_n}{\gamma_n} A_1 \Phi_n(z_1) K_n(z_1; z). \end{aligned} \tag{17}$$

Letting $z = z_1$ in (17) gives

$$\frac{\kappa_n}{\gamma_n}\Phi_n(z_1) - \varphi_n(z_1) = -\frac{\kappa_n}{\gamma_n} A_1 \Phi_n(z_1) K_n(z_1; z_1).$$

Solving for $\Phi_n(z_1)$, and then plugging the result into (17), we can obtain

$$\frac{\kappa_n}{\gamma_n}\frac{\Phi_n(z)}{\varphi_n(z)} = 1 - \frac{A_1 \varphi_n(z_1) K_n(z_1; z)}{\varphi_n(z)(1 + A_1 K_n(z_1; z_1))}. \tag{18}$$

Using the Christoffel–Darboux formula and Lemma 1, the numerator of the fraction on the right side can be written as $A_1|\varphi_n(z_1)|^2\varphi_n(z)(\overline{z_1}z - 1)^{-1}(1 + o(1))$ locally uniformly for $|z| > 1$ as $n \to \infty$. So

$$\frac{\kappa_n}{\gamma_n}\frac{\Phi_n(z)}{\varphi_n(z)} = 1 - \frac{A_1|\varphi_n(z_1)|^2}{1 + A_1K_n(z_1; z_1)}\frac{1 + o(1)}{\overline{z_1}z - 1}.$$

From Thm. 4 in [11], we know that $|\varphi_n(z_1)|^2/K_n(z_1; z_1) \to 0$ as $n \to \infty$ since $z_1 \in \Gamma$. Thus

$$\lim_{n\to\infty}\frac{\kappa_n}{\gamma_n}\frac{\Phi_n(z)}{\varphi_n(z)} = 1,$$

locally uniformly for $|z| > 1$. Letting $z \to \infty$ in the above equation, we get $\lim_{n\to\infty}\kappa_n/\gamma_n = 1$. Thus $\lim_{n\to\infty}\Phi_n(z)/\varphi_n(z) = 1$, locally uniformly for $|z| > 1$.

Now note that measure ν is supported on Γ in this case. We claim $\nu \in N$. In fact, setting $z = 0$ in (18), using the Christoffel–Darboux formula, and then rearranging the terms, we can write

$$\frac{\Phi_n(0)}{\gamma_n} = \frac{\varphi_n(0)}{\kappa_n} - \frac{A_1|\varphi_n(z_1)|^2}{1 + A_1K_n(z_1; z_1)}\left(\overline{\left(\frac{\varphi_n^*(z_1)}{\varphi_n(z_1)}\right)} - \frac{\varphi_n(0)}{\kappa_n}\right).$$

Using the facts that $|\varphi_n(z_1)|^2/K_n(z_1; z_1) \to 0$ as $n \to \infty$ and $|\varphi_n^*(z_1)/\varphi_n(z_1)| = 1$, we see that $\lim_{n\to\infty}\Phi_n(0)/\gamma_n = 0$, which verifies our claim.

The general result now follows from an induction argument based on the previous paragraph and Theorems 1 and 2. We omit the details since the argument is straightforward. ∎

5. Related Results

Suppose μ belongs to N throughout this section. We now discuss some properties of $\Phi_n(z)$ implied by the main results.

First, from the general result on location of zeros of orthogonal polynomials (see, for example, [16, Thm. 2.2]) we know that all zeros of $\Phi_n(z)$ lie in the interior of the convex hull of $\Gamma \cup \{z_1, z_2, \ldots, z_m\}$. Using the main results in Section 2, we can say a little more specifically about the location of zeros of $\Phi_n(z)$.

Proposition 1. *For all n large enough, $\Phi_n(z)$ has exactly m zeros outside Γ tending, respectively, to $z_1, z_2, \ldots, z_m$ from the interior of the* *convex hull of* $\Gamma \cup \{z_1, z_2, \dots, z_m\}$, *and the other* $n - m$ *zeros all lie in the interior of* Γ.

Proof. Note that $\lambda B(z)$ has no zeros in $|z| \geq 1$ except at $\{z_1, z_2, \dots, z_m\}$. If there exists an infinite set Λ of positive integers such that for each $n \in \Lambda$, $\Phi_n(z)$ has at least $m + 1$ zeros in $|z| \geq 1$, then in view of Theorem 2 and Hurwitz's theorem, exactly m of them tend respectively to $z_1, z_2, \dots, z_m$. Then the rest, being contained in the bounded set (the convex hull of $\Gamma \cup \{z_1, z_2, \dots, z_m\}$), must yield at least one limit point, say ξ. Point ξ must be different from z_j, $j = 1, 2, \dots, m$, and $|\xi| \geq 1$. Now Theorem 2 would imply $\lambda B(\xi) = 0$, a contradiction. ∎

What can we say about the distribution of those zeros of $\Phi_n(z)$ lying inside Γ? The answer generally depends on the size of $|z_j|^{-1}$, $j = 1, 2, \dots, m$, and $\rho := \limsup_{n \to \infty} |\varphi_n(0)/\kappa_n|^{1/n}$. We prove one special result on this in the following. The general case remains open. We need to introduce the so called Szegő function associated with measure μ: Let $\mu'(\theta)\, d\theta$ denote the absolute continuous part of measure $d\mu(\theta)$, then, if $\int \log \mu'(\theta)\, d\theta > -\infty$,

$$D(z) := \exp\left\{ \frac{1}{4\pi} \int_{-1\pi}^{\pi} \log \mu'(\theta) \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}}\, d\theta \right\}, \qquad |z| < 1,$$

is called the Szegő function associated with μ. It is well known that (see, for example, [4, Sect. 22]) $\sum_{n=0}^{\infty} |\varphi_n(0)/\kappa_n|^2 < \infty$ implies that the Szegő function exists. So $\rho < 1$ guarantees the existence of $D(z)$. Furthermore, it is proved in [14, Thm. 1] that $\rho < 1$ implies (a) $\{D(z)\}^{-1}$ has an analytic continuation to the disk $|z| < 1/\rho$, (b) $\kappa_n \to \kappa \neq 0$ as $n \to \infty$, and (c) $\varphi_n^*(z)$ is uniformly bounded on $|z| = 1/r$ for every $r > \rho$. These facts will be used in the proof of the following proposition, and we will refer them as fact (a), fact (b) and fact (c), respectively.

Denote $\tau := \max_{1 \leq j \leq m} \{|z_j|^{-1}\}$.

PROPOSITION 2. (i) *For each* $\sigma > \max\{\tau, \rho\}$, *the number of zeros of* Φ_n *lying in* $|z| \geq \sigma$ *is bounded independently of* n.

(ii) *Assume* $\rho < \tau < 1$, *and* $|z_1| = \tau^{-1} < |z_j|$, $j = 2, \dots, m$. *If* $D(z_1) \neq \infty$, *then there is a subsequence of positive integers* Λ *such that, in the weak star topology,*

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} \omega(\Phi_n) = \omega_\tau,$$

where measure $\omega(\Phi_n)$ *is the zero counting unit measure of polynomial* Φ_n *and* ω_τ *is the uniform measure* $d\theta/(2\pi)$ *on the circle* $|z| = \tau$.

Proof. From (15), using the *-transform we can write

$$\Phi_n^*(z) = \frac{\gamma_n}{\kappa_n}\varphi_n^*(z) - \sum_{j=1}^{m} A_j\overline{\Phi_n(z_j)}\varphi_n(z_j)$$
$$\times\left[\left(\frac{\varphi_n^*(z_j)}{\varphi_n(z_j)}\right)\varphi_n(z) - \varphi_n^*(z)\right]\frac{z}{z_j - z}. \qquad (19)$$

Note that, when $\mu \in N$, the ratio asymptotics (7) implies

$$\lim_{n\to\infty} |\varphi(z)|^{1/n} = |z|, \qquad (20)$$

uniformly for $|z| \geq 1$. If $|z_j| < 1/\rho$, then $\varphi_n^*(z_j)$ is bounded by fact (c), and so (20) gives

$$\limsup_{n\to\infty}\left|\frac{\varphi_n^*(z_j)}{\varphi_n(z_j)}\varphi_n(z)\right|^{1/n} \leq \left|\frac{z}{z_j}\right|, \qquad (21)$$

uniformly for $|z| \geq 1$. If $|z_j| \geq 1/\rho$, then, for every $r > \rho$, the Bernstein inequality would yield $|\varphi_n^*(z_j)| \leq (r|z_j|)^n \max_{|z|=1/r}|\varphi_n^*(z)|$, and so by fact (c) and (20)

$$\limsup_{n\to\infty}\left|\frac{\varphi_n^*(z_j)}{\varphi_n(z_j)}\varphi_n(z)\right|^{1/n} \leq |rz|, \qquad (22)$$

uniformly for $|z| \geq 1$.

Hence, for $j = 1, 2, \ldots, m$, we always have

$$\limsup_{n\to\infty}\left|\frac{\varphi_n^*(z_j)}{\varphi_n(z_j)}\varphi_n(z)\right|^{1/n} < 1,$$

locally uniformly for $1 \leq |z| < 1/\tau$. So, with (14) and fact (c), we see (19) implies that $\Phi_n^*(z)$ is locally bounded in $|z| < 1/\tau$. But $\Phi_n^*(z)$ converges to $\lambda B(z)\{D(z)\}^{-1}$ in $|z| \leq 1$. This is because of Theorem 2 and the fact that $\varphi_n^*(z)$ converges to $\{D(z)\}^{-1}$ (cf. [14, Thm. 1]). So $\Phi_n^*(z)$ converges to $\lambda B(z)\{D(z)\}^{-1}$ locally uniformly in $|z| < 1/\tau$. For each compact subset A in $|z| < 1/\tau$, $\lambda B(z)\{D(z)\}^{-1}$ has only finitely many zeros in A, so, by Hurwitz's theorem, the number of zeros of $\Phi_n^*(z)$ in A is bounded independent of n. So part (i) of the proposition is established.

Note that, with fact (c), (14), (21), and (22), Eq. (19) also yields

$$\limsup_{n\to\infty} |\Phi_n^*(z)|^{1/n} \le 1, \tag{23}$$

uniformly for $|z| = 1/\tau$.

Next, we claim that

$$\limsup_{n\to\infty} |\Phi_n(0)|^{1/n} = \frac{1}{|z_1|}. \tag{24}$$

Indeed, letting $z = 0$ in (15) gives

$$\frac{\Phi_n(0)}{\gamma_n} = \frac{\varphi_n(0)}{\kappa_n}\left(1 + \sum_{j=1}^{m} \frac{\kappa_n}{\gamma_n} A_j \Phi_n(z_j)\overline{\varphi_n(z_j)}\right) - \sum_{j=1}^{m} \frac{\kappa_n}{\gamma_n} A_j \Phi_n(z_j)\overline{\varphi_n(z_j)}\overline{\left(\frac{\varphi_n^*(z_j)}{\varphi_n(z_j)}\right)}. \tag{25}$$

As above, we can show that

$$\limsup_{n\to\infty} \left|\frac{\varphi_n^*(z_j)}{\varphi_n(z_j)}\right|^{1/n} \le \rho, \tag{26}$$

if $|z_j| \ge 1/\rho$; and

$$\limsup_{n\to\infty} \left|\frac{\varphi_n^*(z_j)}{\varphi_n(z_j)}\right|^{1/n} \le \frac{1}{|z_j|}, \tag{27}$$

if $|z_j| < 1/\rho$. In fact, we now prove that, when $j = 1$, the equality holds in (27). We need only to show $\limsup_{n\to\infty}|\varphi_n^*(z_1)|^{1/n} = 1$. From the uniform boundedness of $\varphi_n^*(z)$ on $|z| = |z_1|$, it follows that $\limsup_{n\to\infty}|\varphi_n^*(z_1)|^{1/n} \le 1$. Suppose, to the contrary, that the lim sup is strictly less than 1. Then, z_1 must be a zero of the limit function $\lim_{n\to\infty} \varphi_n^*(z)$ that is equal to $\{D(z)\}^{-1}$ (cf. [14, Thm. 1]), contradicting to the hypothesis of the theorem. Now, using the assumption that z_1 is the only point among $\{z_1, z_2, \ldots, z_m\}$ satisfying $|z_1| = 1/\tau < 1/\rho$, and from Theorem 1, (14) in the proof of Theorem 1, (25), (26), and (27), we can write

$$\left|\frac{\Phi_n(0)}{\gamma_n}\right| = \left|\left(\frac{r_1}{B(0)} + o(1)\right)\frac{\varphi_n^*(z_1)}{\varphi_n(z_1)}\right| + o(\rho^n), \tag{28}$$

as $n \to \infty$. As $r_1 \ne 0$ (Lemma 5), together with the remark on the validity of equality in (27) for $j = 1$, we conclude that

$$\limsup_{n\to\infty}\left|\frac{\Phi_n(0)}{\gamma_n}\right|^{1/n} = \frac{1}{|z_1|}. \tag{29}$$

Since $\gamma_n = (\gamma_n/\kappa_n)\kappa_n \to |B(0)|^{-1}\kappa \neq 0$ as $n \to \infty$ (Theorem 1 and fact (b)), we see that $\lim_{n\to\infty} \gamma_n^{1/n} = 1$, so (29) implies our claim (24).

Finally, let Λ be a subsequence of positive integers such that $\lim_{n\to\infty, n\in\Lambda} |\Phi_n(0)|^{1/n} = 1/|z_1|$, and consider $P_n(z) := \Phi_n^*(z)/\overline{\Phi_n(0)}$. In view of (23) and (24),

$$\limsup_{\substack{n\to\infty\\ n\in\Lambda}}\left(\max_{|z|=1/\tau}|P_n(z)|\right)^{1/n} \leq 1/\tau. \tag{30}$$

This, together with part (i) just proved, verifies that for the sequence of monic polynomials $\{P_n(z)\}_{n\in\Lambda}$ and $K = \{z|\ |z| = 1/\tau\}$ the hypotheses of Lemma 3.1 in [12] are satisfied. From this, part (ii) of our proposition then follows. ∎

Using a similar argument, one can prove the following result.

PROPOSITION 3. *If $\rho < 1$ and $|z_1| = 1 < |z_j|$, $j = 2, \ldots, m$, then, in the weak star topology,*

$$\lim_{n\to\infty} \omega(\Phi_n) = \omega_1.$$

Proof. First, by Theorem 2, $\lim_{n\to\infty}|\Phi_n(z)/\varphi_n(z)| = 1$, uniformly on Γ. Now, since $\lim_{n\to\infty}|\varphi_n(z)|^{1/n} = 1$ uniformly on Γ, we get $\lim_{n\to\infty}|\Phi_n(z)|^{1/n} = 1$ or

$$\lim_{n\to\infty}|\Phi_n^*(z)|^{1/n} = 1, \tag{31}$$

uniformly on Γ. Next, we show that $\lim_{n\to\infty}|\Phi_n(0)|^{1/n} = 1$. The key is that the equality in (27) for $j = 1$ holds when $|z_1| = 1$ without any condition on the Szegő function. In fact, $|\varphi_n^*(z)/\varphi_n(z)| = 1$ on Γ, so we have $\lim_{n\to\infty}|\varphi_n^*(z_1)/\varphi_n(z_1)|^{1/n} = 1$. With this in the place of (27) for $j = 1$, we can proceed as in the proof of part (ii) of Proposition 2 to show $P_n(z) := \Phi_n^*(z)/\overline{\Phi_n(0)}$ satisfies

$$\lim_{n\to\infty}|P_n(z)|^{1/n} = 1,$$

uniformly on Γ. This, together with part (i) of Proposition 2, verifies all hypotheses of Lemma 3.1 in [12] with $\{P_n(z)\}$ and $K = \Gamma$. The proof is complete. ∎

It is interesting to notice the implication of Proposition 3 for the zeros of polynomials orthogonal on Γ (cf. [12, 14, 16]). For the uniform measure $d\theta/(2\pi)$ on Γ, all zeros of its orthogonal polynomial z^n lie at $z = 0$. Proposition 3 tells us that by adding one mass point, say $d\delta_0(\theta)$, the majority of the zeros of orthogonal polynomial will be attracted to Γ. This is the case since the nth monic orthogonal polynomial corresponding to $d\theta/(2\pi) + d\delta_0(\theta)$ is given by (cf. (18))

$$z^n - \frac{1 + z + z^2 + \cdots + z^{n-1}}{n+1},$$

and it is straightforward to show that all zeros of this polynomial are contained in $|z| \geq (2n+3)^{-1/n}$. This zero distribution can also be verified by Thm. 2.3 in [12].

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